MATHEMATICS

DERIVED MARKOV CHAINS. I

BY

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1. *Introduction*

In this study we describe a method by which from a given stationary Markov chain ${}_1M$ a new Markov chain ${}_2M$ is derived; the Markov chain ${}_2M$ is called a Markov chain derived from ${}_1M$. Both chains have the same state space. A Markov chain derived from ${}_1M$ is completely defined by ${}_1M$ and a function $b(s,t)$, satisfying some conditions, of two real variables s and t with $t \in (-\infty, \infty)$ and $s \in [0, \infty)$ or $s = nt_0$, $n = 0, 1, 2, \ldots, t_0$ being a constant. The chain ${}_1M$ may be a discrete or a continuous parameter chain and similar ${}_2M$ may be a discrete or a continuous parameter chain, so that with respect to the type of the timeparameter four combinations of ${}_1M$ and ${}_2M$ are possible.

In this study it is always supposed that the chains have a discrete state space; however, generalizations for nondiscrete state space can be easily performed.

The first part of the investigations is devoted to the definition of the concept of derived Markov chains and to the study of the properties of the deriving function $b(.,.)$. The results show that the process of deriving Markov chains from a given Markov chain is closely related to the process of generating semigroups of linear operators (cf. PHILLIPS, [3]) in the case that ${}_1M$ and ${}_2M$ both have a continuous timeparameter.

In the second part of the study the relations between properties of ${}_2M$ and ${}_1M$ are described for all possible combinations of the types of the time-parameters of ${}_1M$ and ${}_2M$.

The results obtained in this study describe only some of the relations between a Markov chain and a chain derived from it, and undoubtedly many other properties can be found. It should be noted that an important problem is to find conditions which guarantee that a given Markov chain is a derived Markov chain. Derived Markov chains are also of great practical interest since a general method for solving important queueing- and inventory problems has been obtained by applying the theory of derived Markov chains. About these practical applications we shall communicate in a seperate paper.

2. *Definition of "derived Markov chain"*

Since we want to treat simultaneously Markov chains with a continuous timeparameter (c.p.) and with a discrete timeparameter (d.p.) at least as far as possible it is desirable to introduce symbols for the various sets of values taken on by the timeparameter t or s. Therefore, we define

$$T_c^{0+} \stackrel{\mathrm{df}}{=} \{t:\ 0<t<\infty\}, \qquad T_d^{0+} \stackrel{\mathrm{df}}{=} \{t:\ t=nt_0,\quad n \in N_1\},$$
$$T_c^{0} \stackrel{\mathrm{df}}{=} \{t:\ 0\leqq t<\infty\}, \qquad T_d^{0} \stackrel{\mathrm{df}}{=} \{t:\ t=nt_0,\quad n \in N_0\},$$
$$T_c^{-\infty} \stackrel{\mathrm{df}}{=} \{t:\ -\infty<t<0\} \cup T_c^0, T_d^{-\infty} \stackrel{\mathrm{df}}{=} \{t:\ -\infty<t<0\} \cup T_d^0,$$

where t_0 is a fixed positive constant, and N_i is the set of integers not less than i, i being an integer. Whenever the symbol $T^{\cdot}$ is written without a subscript it stands for $T_c^{\cdot}$ or $T_d^{\cdot}$.

By $\mathscr{N}$ will be denoted a denumerable set of elements, and a generic element of $\mathscr{N}$ will be represented by i, j or h.

We next introduce the matrix

$$_1P(t) \equiv (_1p_{ij}(t)), \quad i, j \in\in \mathscr{N},$$

where for all $i, j \in\in \mathscr{N}$ the $_1p_{ij}(.)$ are functions defined on $T^{-\infty}$ such that

V_1, $\qquad _1P(t)=I$ for all $t \in (-\infty, 0]$,

and for all t, $\tau \in\in T^{0+}$

V_2, $\qquad _1p_{ij}(t) \geqq 0$,

V_3, $\qquad \sum_{h \in \mathscr{N}} {_1p_{ih}(t)} = 1$,

V_4, $\qquad _1P(t)\,_1P(\tau) = {_1P(t+\tau)}$;

moreover whenever $_1P(t)$ is defined for all $t \in T_c^{0+}$ it is assumed that

V_5, $\qquad \lim_{t \downarrow 0} {_1P(t)} = I.$

Here I denotes the unit matrix. In V_5 the limit operation on the matrix should be read as the limit of element per element of the matrix. Unless stated otherwise, all operations on any matrix are to be interpreted as operations on element per element of the matrix. It should be remarked that the assumption V_1 is not essential; it is introduced for convenience of notation.

By E_i, $i \in \mathscr{N}$ will be denoted a state, and the set of all states is indicated by $\mathscr{E}$; $p(.)$ will represent a probability distribution defined on $\mathscr{E}$.

Whenever $_1P(t)$ is defined on T_d^{0+} then by V_4

$$_1P(t) \equiv {_1P(nt_0)} = {_1P^n(t_0)}, \quad n \in N_1,$$

and $_1P(nt_0)$ represents for every fixed $n \in N_1$ a stochastic matrix since all its elements are nonnegative (V_2) and all its row sums are equal to one (V_3). From a wellknown theorem of d.p. Markov chains (cf. [1] p. 7) it follows that a (stationary) d.p. Markov chain $_1M$ exists which has $\mathscr{E}$ as state space, $p(.)$ as initial distribution and $_1P(nt_0)$ as n-step transition matrix.

Whenever ${}_1P(t)$ is defined for $t \in T_c{}^0$ then it follows from V_2, V_3 and V_4 that ${}_1P(t)$ is a transition matrix. Hence (cf. [1] p. 137), a (stationary) c.p. Markov chain ${}_1M$ exists with $\mathscr{E}$ as minimal state space, with $p(.)$ as initial distribution and with ${}_1P(t)$ as transition matrix. The assumption V_5 implies that ${}_1P(t)$ is a standard transition matrix, and consequently it follows that

c.i. for all $i, j \in\in \mathscr{N}$ the functions ${}_1p_{ij}(.)$ are uniformly continuous in $T_c{}^0$ (cf. [1] p. 124);

c.ii. for all $i, j \in\in \mathscr{N}$ the function ${}_1p_{ij}(.)$, $i \neq j$, is either identically zero or never zero in $T_c{}^{0+}$, and ${}_1p_{ii}(.)$ is never zero in $T_c{}^{0+}$ (cf. [1] p. 121.);

c.iii. the following limits exist always,

$$(2.1) \qquad {}_1Q \equiv ({}_1q_{ij}) \overset{\text{df}}{=} \lim_{t \downarrow 0} \frac{{}_1P(t) - I}{t},$$

in the sense that all ${}_1q_{ij}$, $i \neq j$, are finite, whereas ${}_1q_{ii}$ may be infinite (cf. [1] p. 126, 127).

We now introduce a function $b(.,.)$ defined on $T^0 \times T_c{}^{-\infty}$ which shall have the following properties:

V_6, for every fixed $s \in T^{0+}$ the function $b(s,.)$ is a probability distribution, continuous from the left, of a nonnegative stochastic variable;

$$V_7, \qquad b(0, t) = U(t) \text{ for all } t \in T_c{}^{-\infty};$$

and whenever $b(.,.)$ is defined on $T_c{}^0 \times T_c{}^{-\infty}$ then $b(s,.)$ converges completely to $U(.)$ for $s \downarrow 0$; here $U(.)$ denotes the unit step function, i.e.

$$U(t) = 0 \text{ for } t \leqq 0, \quad U(t) = 1 \text{ for } t > 0;$$

V_8, for every fixed $t \in T_c{}^{-\infty}$ and all $s, \sigma \in\in T^0$

$$b(s, t) * b(\sigma, t) = b(s + \sigma, t),$$

where "$*$" denotes the convolution operation.

Evidently, two types of functions are defined above, viz. functions defined on $T_c{}^0 \times T_c{}^{-\infty}$ and functions defined on $T_d{}^0 \times T_c{}^{-\infty}$. It is easily verified that the assumptions V_6, V_7 and V_8 are compatible.

Let $b(.,.)$ be a function as introduced above; we define

$$(2.2) \qquad \mathscr{E}_s\{{}_1P(t)\} \overset{\text{df}}{=} \int_{-\infty}^{+\infty} {}_1P(t)\, d_t b(s, t), \quad s \in T^0,$$

whenever ${}_1P(.)$ is defined on $T_c{}^{-\infty}$;

$$(2.3) \qquad \mathscr{E}_s\{{}_1P(t)\} \overset{\text{df}}{=} \sum_{n=0}^{\infty} {}_1P(nt_0)\, \Delta b(s, nt_0), \quad s \in T^0,$$

whenever: i. ${}_1P(.)$ is defined on $T_d{}^{-\infty}$ and ii. for every fixed $s \in T^0$ the function $b(s,.)$ is a (pure) stepfunction of which the set of discontinuities is contained in $T_d^{-\infty}$; here $\Delta b(s, nt_0)$ denotes the saltus of $b(s,.)$ in $t = nt_0$. By $V_1, \ldots, V_7$ and c.i. it is easily seen that the righthand sides of (2.2) and (2.3) always exist.

Putting

$$(2.4) \qquad {}_2P(s) \equiv ({}_2p_{ij}(s)) \overset{\mathrm{df}}{=} \mathscr{E}_s\{{}_1P(t)\}, \quad s \in T^0,$$

then by V_2, V_6, (2.2) and (2.3) for all $i, j \in\in \mathscr{N}$ and all $s \in T^{0+}$

$$(2.5) \qquad 1 \geqq {}_2p_{ij}(s) \geqq 0;$$

moreover by V_2, V_3, V_6 and the Fubini theorem we have for all $i \in \mathscr{N}$ and all $s \in T^{0+}$

$$(2.6) \qquad \sum_{h \in \mathscr{N}} {}_2p_{ih}(s) = \sum_{h \in \mathscr{N}} \mathscr{E}_s\{{}_1p_{ih}(t)\} = \mathscr{E}_s\{\sum_{h \in \mathscr{N}} {}_1p_{ih}(t)\} = \mathscr{E}_s\{1\} = 1.$$

Further

$$(2.7) \qquad {}_2P(s) = I \text{ for } s = 0,$$

and whenever ${}_2P(s)$ is defined for all $s \in T_c^0$

$$(2.8) \qquad \lim_{s \downarrow 0} {}_2P(s) = I.$$

Relation (2.7) is immediately evident. To prove (2.8) note that for all $i, j \in\in \mathscr{N}$ the function ${}_1p_{ij}(.)$ is a bounded continuous function on $T_c^{-\infty}$ (cf. V_1 and c.i.) and that $b(s,.)$ converges completely to $U(.)$ for $s \downarrow 0$ (cf. V_7); hence, application of the Helly–Bray theorem (cf. [2] p. 182) proves (2.8).

Consider for fixed $s \in T^0$ the function $b(s,.)$ as the probability distribution of a stochastic variable $\mathbf{t}$, then by (2.2) or (2.3) and c.i. ${}_2p_{ij}(s)$ may be regarded as the mathematical expectation of the stochastic variable ${}_1p_{ij}(\mathbf{t})$. Generally, ${}_2P(s)$ may be regarded as the mathematical expectation of the matrix ${}_1P(\mathbf{t})$ with respect to the distribution $b(s,.)$. Similarly, let $\boldsymbol{\tau}$ be a stochastic variable with distribution $b(\sigma,.)$, $\sigma \in T^0$, and be ${}_2P(\sigma) = \mathscr{E}_\sigma\{{}_1P(\boldsymbol{\tau})\}$. We may and do assume that $\mathbf{t}$ and $\boldsymbol{\tau}$ are independent variables. Hence, for all pairs $i_1, j_1 \in\in \mathscr{N}$ and $i_2, j_2 \in\in \mathscr{N}$ the stochastic variables ${}_1p_{i_1j_1}(\mathbf{t})$ and ${}_1p_{i_2j_2}(\boldsymbol{\tau})$ are independent, since they are finite Borel functions of independent stochastic variables. By (2.5) $\mathscr{E}_s\{{}_1p_{ih}(\mathbf{t})\}$ and $\mathscr{E}_\sigma\{{}_1p_{hj}(\boldsymbol{\tau})\}$ are finite for all $i, j, h \in\in \mathscr{N}$ so that by independence of $\mathbf{t}$ and $\boldsymbol{\tau}$

$$\mathscr{E}_s\{{}_1p_{ih}(\mathbf{t})\}\,\mathscr{E}_\sigma\{{}_1p_{hj}(\boldsymbol{\tau})\} = \mathscr{E}_{s,\sigma}\{{}_1p_{ih}(\mathbf{t})\,{}_1p_{hj}(\boldsymbol{\tau})\},$$

where the righthand side denotes the mathematical expectation of ${}_1p_{ih}(\mathbf{t})\,{}_1p_{hj}(\boldsymbol{\tau})$ with respect to the joint distribution of $\mathbf{t}$ and $\boldsymbol{\tau}$. By V_2, V_6, (2.6) and Fubini's theorem the relation above implies

$$\begin{aligned} {}_2P(s)\,{}_2P(\sigma) &= \mathscr{E}_s\{{}_1P(\mathbf{t})\}\,\mathscr{E}_\sigma\{{}_1P(\boldsymbol{\tau})\} = \mathscr{E}_{s,\sigma}\{{}_1P(\mathbf{t})\,{}_1P(\boldsymbol{\tau})\} \\ &= \mathscr{E}_{s,\sigma}\{{}_1P(\mathbf{t}+\boldsymbol{\tau})\} \text{ by } V_4. \end{aligned}$$

Since $\mathbf{t}$ and $\boldsymbol{\tau}$ are independent the distribution of $\mathbf{z} \overset{\mathrm{df}}{=} \mathbf{t}+\boldsymbol{\tau}$ is given by $b(s,.) * b(\sigma,.)$, i.e. by $b(s+\sigma,.)$ according to V_8. Hence, the mathematical expectation $\mathscr{E}_{s,\sigma}\{{}_1P(\mathbf{t}+\boldsymbol{\tau})\}$ of ${}_1P(\mathbf{t}+\boldsymbol{\tau})$ with respect to the joint distribution of $\mathbf{t}$ and $\boldsymbol{\tau}$ is given by (cf. (2.2) or (2.3))

$$\mathscr{E}_{s,\sigma}\{{}_1P(\mathbf{t}+\boldsymbol{\tau})\} = \mathscr{E}_{s+\sigma}\{{}_1P(\mathbf{z})\} = {}_2P(s+\sigma).$$

Consequently, for all $s, \sigma \in\in T^0$

$$(2.9) \qquad {}_2P(s+\sigma) = {}_2P(s)\,{}_2P(\sigma).$$

and hence by (2.5), ..., (2.9) it is seen that ${}_2P(s)$ is for $s \in T_c{}^0$ a standard transition matrix.

Whenever $s \in T_d{}^0$ then the matrix

$${}_2P(s) \equiv {}_2P(nt_0) = {}_2P^n(t_0), \quad n \in N_1,$$

is a stochastic matrix and consequently, a (stationary) d.p. Markov chain ${}_2M$ exists with state space $\mathscr{E}$, with initial distribution $p(.)$ and with ${}_2P(nt_0)$ as n-step transition matrix.

Similarly, whenever $s \in T_c{}^0$ a (stationary) c.p. Markov chain exists with minimal state space $\mathscr{E}$, with initial distribution $p(.)$ and with standard transition matrix ${}_2P(s)$.

Evidently, the following theorem has been proved above.

Theorem 2.1. Whenever ${}_1M$ is a stationary Markov chain with state space $\mathscr{E}$, with initial distribution $p(.)$, and with transition matrix ${}_1P(t)$, $t \in T^{-\infty}$ (statisfying $V_1, \ldots, V_4$ and also V_5 if $t \in T_c{}^{-\infty}$) and $b(.,.)$ is a function defined on $T^0 \times T_c{}^{-\infty}$ satisfying (the relevant parts of) $V_6, \ldots, V_8$, then a stationary Markov chain ${}_2M$ exists with $\mathscr{E}$ as state space, $p(.)$ as initial distribution and with ${}_2P(s)$, $s \in T^0$, defined by (2.4), as transition matrix; this matrix is standard if $s \in T_c{}^0$.

A chain ${}_2M$ described as in the theorem above will be called a Markov chain derived from ${}_1M$ by $b(.,.)$. By the statement "a chain M is a derived Markov chain" will be meant that there exists a Markov chain ${}_1M$ and a function $b(.,.)$ with properties as mentioned in theorem 2.1 such that M is a Markov chain derived from ${}_1M$ by $b(.,.)$; ${}_1M$ will be called the original chain and $b(.,.)$ the deriving function.

Since a Markov chain with the unit matrix I as transition matrix is rather uninteresting we shall assume from now on that

$$(2.10) \qquad b(s,.) \neq U(.) \text{ for a } s \in T^{0+},$$

i.e. we exclude (cf. V_8 and (2.9)) the possibility that ${}_2P(s) = I$ for a, and hence for all, $s \in T^{0+}$.

3. *On the functions* $b(.,.)$

Before investigating relations between a Markov chain and a chain derived from it, it is necessary to study the properties of deriving functions.

We shall denote by $\mathscr{G}_d(\mathscr{G}_c)$ the class of deriving functions $b(.,.)$ defined on $T_d{}^0 \times T_c{}^{-\infty}$ $(T_c{}^0 \times T_c{}^{-\infty})$.

Consider first the class $\mathscr{G}_d$. In this case the convolution property V_8 may be rewritten as

$$(3.1) \qquad b(nt_0, t) * b(mt_0, t) = b((m+n)t_0, t), \;\; m, n \in\in N_0, \;\; t \in T_c{}^{-\infty}.$$

Putting

$$(3.2) \qquad a^{0*}(t) \overset{\mathrm{df}}{=} U(t), \;\; a(t) \overset{\mathrm{df}}{=} b(t_0, t), \;\; t \in T_c{}^{-\infty},$$

then by V_7, V_8 and (3.1)

$$(3.3) \qquad b(s, t) = b(nt_0, t) = a^{n*}(t) \text{ for all } n \in N_0, \;\; t \in T_c{}^{-\infty},$$

where $a^{n*}(.)$, $n \in N_1$ denotes the n-fold convolution of $a(.)$ with itself. Evidently, $a^{n*}(.)$ represents for every $n \in N_0$ a probability distribution of a nonnegative stochastic variable. It is noted that an element $b(.,.) \in \mathscr{G}_d$ is completely determined whenever $b(s,.)$ is known for one value of $s \in T_d{}^{0+}$.

Consider next the class $\mathscr{G}_c$. As already stated an element $b(.,.)$ of $\mathscr{G}_c$ represents for every fixed $s \in T_c{}^0$ a probability distribution of a nonnegative stochastic variable. Hence, if $b(.,.) \in \mathscr{G}_c$ then V_8 implies that $b(s,.)$ represents for every fixed $s \in T_c{}^0$ an infinitely divisible distribution of a nonnegative stochastic variable. Such distributions have been studied by PHILLIPS (cf. [3] and [4] p. 660).

Putting

$$(3.4) \qquad \beta(s, \lambda) \overset{\mathrm{df}}{=} \int_{-\infty}^{+\infty} e^{-\lambda t}\, d_t b(s, t), \;\; \mathrm{Re}\, \lambda \geqq 0, \;\; s \in T_c{}^0,$$

it follows from Phillips' results that for every $b(.,.) \in \mathscr{G}_c$ the logarithm of its Laplace–Stieltjes transform $\beta(s, \lambda)$ can be written as

$$(3.5) \qquad \log \beta(s, \lambda) = s\left[-m_0\lambda + \int_0^\infty \left\{e^{-\lambda t} - 1 + \frac{\lambda t}{1+t^2}\right\} \frac{1+t^2}{t^2}\, dG(t)\right], \;\; s \in T_c{}^0,$$

where m_0 is a real number and $G(.)$ is a real bounded, nondecreasing function with

$$(3.6) \qquad \begin{cases} G(t) = 0 \text{ for } t \in (-\infty, 0], \;\; G(0+) = 0, \\ m_0 \geqq \int_0^\infty t^{-1}\, dG(t) \geqq 0; \end{cases}$$

m_0 and $G(.)$ are uniquely determined by $b(s,.)$.

Putting

$$(3.7) \qquad \begin{cases} \Psi(t) \overset{\mathrm{df}}{=} -\int_t^\infty \dfrac{1+x^2}{x^2}\, dG(x), \;\; t \in (0, \infty), \\ m \overset{\mathrm{df}}{=} m_0 - \int_0^\infty t^{-1}\, dG(t), \end{cases}$$

then

$$(3.8)\qquad \begin{cases} \Psi(t) \leqq 0 \text{ for } t \in (0, \infty), \int_0^1 t d\Psi(t) < \infty, \ \Psi(\infty) = 0, \ m \geqq 0, \\ \Psi(t_1) \leqq \Psi(t_2) \text{ for all } t_1, \ t_2 \in\in (0, \infty) \text{ with } t_1 \leqq t_2; \end{cases}$$

and from (3.5)

$$(3.9)\qquad \log \beta(s, \lambda) = s\{-m\lambda + \int_0^\infty (e^{-\lambda t} - 1) d\, \Psi(t), \ \mathrm{Re}\, \lambda \geqq 0, \ s \in T_c{}^0.$$

Here m and $\Psi(.)$ are uniquely determined by $b(s,.)$, and conversely, every $m \geqq 0$ and every function $\Psi(.)$ on $(0, \infty)$ satisfying (3.8) determine uniquely by (3.9) and (3.4) a function $b(.,.) \in \mathscr{G}_c$.

Next we recall a theorem due to BLUM and ROSENBLATT [5]. Denoting by $\varphi(u)$ the characteristic function of an infinitely divisible distribution $F(.)$ on $(-\infty, \infty)$ then

$$\log \varphi(u) = i\gamma u + \int_{-\infty}^{+\infty} \left\{ e^{iut} - 1 - \frac{iut}{1+t^2} \right\} \frac{1+t^2}{t^2} \, dH(t),$$

where γ is a real number and $H(.)$ is a real bounded, nondecreasing function with $H(-\infty) = 0$; and

i. $F(.)$ is a discrete distribution if and only if $H(.)$ is a (pure) step-function and $\int_{-\infty}^{\infty} t^{-2} \, dH(t) < \infty$;

ii. $F(.)$ is a mixed distribution if and only if $H(.)$ is not a (pure) step-function and $\int_{-\infty}^{\infty} t^{-2} \, dH(t) < \infty$;

iii. $F(.)$ is a continuous distribution if and only if $\int_{-\infty}^{+\infty} t^{-2} dH(t) = \infty$.

Applying this theorem to the function $b(.,.) \in \mathscr{G}_c$ then for every fixed $s \in T_c{}^{0+}$:

i. $b(s,.)$ is a discrete distribution if and only if $\Psi(.)$ is a stepfunction and $\Psi(0+) > -\infty$;

ii. $b(s,.)$ is a mixed distribution if and only if $\Psi(.)$ is not a stepfunction and $\Psi(0+) > -\infty$;

iii. $b(s,.)$ is a continuous distribution if and only if $\Psi(0+) = -\infty$.

Let us consider in some detail the case $\Psi(0+) > -\infty$. We may then define

$$(3.10)\qquad \begin{cases} b(t) \overset{\mathrm{df}}{=} 1 - \dfrac{\Psi(t)}{\Psi(0+)} & \text{for } t \in (0, \infty), \\ \overset{\mathrm{df}}{=} 0 & \text{for } t \in (-\infty, 0], \end{cases}$$

so that $b(0+) = 0$; $b(.)$ may be regarded as a distribution function of a nonnegative stochastic variable. From (3.9) it now follows

$$(3.11)\quad \begin{cases} \beta(s,\lambda) = \exp\{-m\lambda s + s\Psi(0+)\,(1-\int\limits_0^\infty e^{-\lambda t}\,db(t))\}, \quad \operatorname{Re}\lambda \geqq 0, \\ \quad = e^{-m\lambda s+s\Psi(0+)} \sum\limits_{n=0}^{\infty} \dfrac{\{-s\Psi(0+)\}^n}{n!} \{\int\limits_0^\infty e^{-\lambda t}\,db(t)\}^n. \end{cases}$$

Since for all $s \in T_c{}^0$ the inverse of the Laplace–Stieltjes transform $\beta(s,\lambda)$ is uniquely determined if it should satisfy the relevant part of V_6, we have by inverting the righthand member of (3.11) term by term, which is evidently allowed,

$$(3.12)\quad b(s,t) = e^{s\Psi(0+)} \sum_{n=0}^{\infty} \frac{\{-s\Psi(0+)\}^n}{n!}\, b^{n*}(t-ms), \quad s \in T_c{}^0,\ t \in T_c{}^{-\infty},$$

where $b^{0*}(.) \stackrel{\text{df}}{=} U(.)$ and $b^{n*}(.)$ is the n-fold convolution of $b(.)$ with itself, $n \in N_1$.

As already stated the deriving function $b(.,.)$ must be for every fixed $s \in T_c{}^0$ a (pure) stepfunction with points of discontinuity all belonging to $T_d{}^0$ whenever the original chain ${}_1M$ is a d.p. chain. According to the results of Blum and Rosenblatt in this case $\Psi(0+)$ should be finite and $\Psi(.)$, and hence $b(.)$ (cf. (3.10)), should be a stepfunction. These conditions guarantee that $b(s,.)$ is for every fixed $s \in T_c{}^0$ a stepfunction. However, if $m > 0$ then (cf. (3.12)) the set of discontinuity points of $b(s,.)$ depends on s. Consequently, the necessary and as it is easily seen, also sufficient conditions for the deriving function $b(.,.) \in \mathscr{G}_c$ to be for every fixed $s \in T_c{}^0$ a stepfunction with its points of increase all belonging to $T_d{}^0$ are $m=0$, $\Psi(0+) > -\infty$ and $\Psi(.)$ is a pure stepfunction with its points of increase all belonging to $T_d{}^{0+}$. The latter two conditions are equivalent with: "$b(.)$ is a discrete probability distribution with points of increase all belonging to $T_d{}^{0+}$". Evidently, the general representation of $b(.,.)$ in this case is given by

$$(3.13)\qquad b(s,t) = e^{s\Psi(0+)} \sum_{n=0}^{\infty} \frac{\{-s\Psi(0+)\}^n}{n!}\, b^{n*}(t), \quad s \in T_c{}^0,\ t \in T_0{}^{-\infty}.$$

Finally, we note that the condition (2.10) is equivalent with

$$(3.14)\qquad \begin{cases} a(.) \neq U(.) & \text{if}\quad b(.,.) \in \mathscr{G}_d, \\ m-\Psi(0+) > 0 & \text{if}\quad b(.,.) \in \mathscr{G}_c, \end{cases}$$

and that for all finite $s \in [0,\infty)$

$$(3.15)\qquad \int\limits_{-\infty}^{+\infty} t\,d_t\,b(s,t) < \infty \Leftrightarrow \int\limits_0^\infty t\,d\Psi(t) < \infty .$$

The relation (3.15) follows easily from (3.9) and a known relation between the first moment of a probability distribution and the derivative at zero of its characteristic function (cf. [2], complements and details, 11. p. 217).

4. *A theorem on skeleton chains*

The concept of "skeleton chain at time scale s_0" has been introduced by CHUNG ([1] p. 127). Let M be a stationary, c.p. Markov chain with state space $\mathscr{E}$ and with transition matrix $P(s)$, $s \in T_c{}^0$ (the trivial case $P(s) \equiv I$ will be excluded). The stationary, d.p. Markov chain Ms_0 with state space $\mathscr{E}$, with same initial distribution as M and with one-step transition matrix $P(s_0)$ has been called by Chung a skeleton chain of M at time scale s_0. Since t_0 is an arbitrary but fixed constant (cf. section 2) we choose in the following t_0 as time scale, i.e. $s_0 = t_0$.

Whenever $P(s)$, $s \in T_c{}^0$ is a standard transition matrix then evidently a skeleton chain of M at time scale t_0 may be also defined as the chain derived from M by the deriving function $U(t-s)$, $s \in T_d{}^0$, since by (2.2)

$$P(s) \equiv P^n(t_0) = \int_{-\infty}^{+\infty} P(t) d_t\, U(t-s), \quad s \in T_d{}^0.$$

Note that for every fixed $s \in T_d{}^{0+}$ the deriving function $U(t-s)$ is an infinitely divisible distribution.

We now prove

Theorem 4.1. A stationary, d.p. Markov chain is a skeleton chain of a c.p. Markov chain with standard transition matrix if it is a derived chain and if the deriving function $b(s, t)$, $s \in T_d{}^0$, $t \in T_c{}^{-\infty}$ is for a fixed $s \in T_d{}^{0+}$ an infinitely divisible distribution of a nonnegative stochastic variable.

Proof. Let ${}_2M^d$ denote a stationary, d.p. Markov chain derived from a Markov chain ${}_1M$ by a function $b(.,.) \in \mathscr{G}_d$, and denote by ${}_2P_d(t_0)$ the one-step transition matrix of ${}_2M^d$. As in (3.2) we write $a^{n*}(.) = b(nt_0, .)$, $n \in N_0$. By hypothesis $a^{n*}(.)$ is for some value n_0 of $n \in N_1$ an infinitely divisible distribution of a nonnegative stochastic variable. Consequently, $a(.)$ is also such a distribution function so that by the results of Phillips (cf. section 3) the Laplace–Stieltjes transform $\alpha(\lambda)$ of $a(.)$ may be expressed as

$$\alpha(\lambda) \stackrel{\text{df}}{=} \int_{-\infty}^{+\infty} e^{-\lambda t}\, da(t)$$

$$= e^{-m\lambda + \int_0^\infty (e^{-\lambda t} - 1)\, d\Psi(t)}, \quad \operatorname{Re} \lambda \geqq 0,$$

where m and $\Psi(.)$ satisfy (3.8) and are completely determined by $a(.)$. Hence, for every fixed $s \in T_c{}^0$

$$\{\alpha(\lambda)\}^s = e^{-m\lambda s + s\int_0^\infty (e^{-\lambda t} - 1)\, d\Psi(t)}, \quad \operatorname{Re} \lambda \geqq 0,$$

is also a Laplace–Stieltjes transform, and it is actually the Laplace–Stieltjes transform of a uniquely determined function $b_0(s, t)$ such that $b_0(.,.) \in \mathscr{G}_c$. Denoting by ${}_2M^c$ the c.p. Markov chain derived from ${}_1M$ by

$b_0(.,.)$ and by ${}_2P_c(s)$, $s \in T_c{}^0$ the transition matrix of ${}_2M^c$, which is standard (cf. theorem 2.1) then by (2.4)

$$\begin{aligned} {}_2P_d{}^n(t_0) &= \mathscr{E}_s\{{}_1P(t)\}, \; s = nt_0, \; n \in N_0, \\ {}_2P_c(s) &= \mathscr{E}_s\{{}_1P(t)\}, \; s \in T_c{}^0, \end{aligned}$$

where the "expectation" is taken with respect to $b_0(s,.)$, and where ${}_1P(.)$ is the transition matrix of ${}_1M$. From the last relation and the definition of skeleton chain it is seen that ${}_2M^d$ is a skeleton chain at time scale t_0 of the stationary, c.p. Markov chain ${}_2M^c$ with standard transition matrix ${}_2P_c(s)$, $s \in T_c{}^0$. This proves the theorem.

5. ${}_1M$ *a c.p. chain,* ${}_2M$ *a d.p. chain*

Whenever the original chain is a c.p. chain and the derived chain a d.p. chain, so that the deriving function $b(.,.) \in \mathscr{G}_d$ then by (2.2), (2.4) and (3.3)

$$(5.1) \qquad {}_2P(s) = {}_2P(nt_0) = {}_2P^n(t_0) = \int_{-\infty}^{+\infty} {}_1P(t)\, da^{n*}(t). \quad s \in T_d{}^0.$$

Theorem 5.1. If ${}_2M$ is a d.p. chain derived from a c.p. chain ${}_1M$ (note (2.10)) then ${}_2M$ is an aperiodical chain and

$$(5.2) \qquad \lim_{s\to\infty} {}_2P(s) = \lim_{n\to\infty} {}_2P(nt_0) = \lim_{t\to\infty} {}_1P(t),$$

moreover:

i. a positive state of ${}_1M$ is a positive state of ${}_2M$ and consersely;

ii. a transient state of ${}_1M$ is a transient state of ${}_2M$;

iii. a null state of ${}_1M$ is a null state of ${}_2M$ if

$$\int_{-\infty}^{+\infty} t da(t) \equiv \int_{-\infty}^{+\infty} t db(t_0, t) < \infty,$$

whereas a null state of ${}_1M$ is a null state or a transient state of ${}_2M$ if this integral diverges.

Proof. Since ${}_1p_{ii}(t) > 0$ for all $t \in T_c{}^0$, all $i \in \mathscr{N}$ (cf. c.ii. section 2) it follows by (5.1) that ${}_2p_{ii}{}^{(n)}(t_0) > 0$ for all $n \in N_0$, all $i \in \mathscr{N}$; this implies that the derived chain ${}_2M$ is aperiodical.

From (3.2) and (3.3) it follows that (cf. 3.14)

$$a^{(n+1)*}(t) = \int_{0-}^{t} a(t-\tau) da^{n*}(\tau) \leqq a^{n*}(t)\, a(t) \leqq a^{n+1}(t),$$

hence

$$(5.3) \qquad \lim_{n\to\infty} a^{n*}(t) = 0 \text{ for every finite } t \in (-\infty, \infty).$$

Since ${}_1M$ is a c.p. Markov chain with standard transition matrix it follows that $\lim_{t\to\infty} {}_1p_{ij}(t)$ exists for all $i, j \in\in \mathscr{N}$ (cf. [1] p. 178). Put

$${}_1\pi_{c,ij} \overset{\text{df}}{=} \lim_{t\to\infty} {}_1p_{ij}(t) \leqq 1, \; i, j \in\in \mathscr{N},$$

hence by (3.2), (5.1) for all $n \in N_0$

$$_2p_{ij}^{(n)}(t_0) - {}_1\pi_{c,ij} = \int_{-\infty}^{+\infty} \{{}_1p_{ij}(t) - {}_1\pi_{c,ij}\}\, da^{n*}(t).$$

For arbitrary $\delta > 0$ a number $T_{ij} \in [0, \infty)$ exists such that

$$|{}_1p_{ij}(t) - {}_1\pi_{c,ij}| < \delta \text{ for all } t > T_{ij},$$

hence

$$|{}_2p_{ij}^{(n)}(t_0) - {}_1\pi_{c,ij}| \leq \int_{-\infty}^{+\infty} |{}_1p_{ij}(t) - {}_1\pi_{c,ij}|\, da^{n*}(t) \leq$$

$$\leq 2 \int_{-\infty}^{T_{ij}} da^{n*}(t) + \delta \int_{T_{ij}}^{\infty} da^{n*}(t) = 2a^{n*}(T_{ij}) + \delta\{1 - a^{n*}(T_{ij})\}.$$

Consequently, by (5.3) for all $i, j \in \in \mathscr{N}$,

$$\lim_{n\to\infty} {}_2p_{ij}^{(n)}(t_0) = {}_1\pi_{c,ij},$$

i.e. (5.2) is proved.

A state $E_j \in \mathscr{E}$ is a positive state of ${}_1M$ if and only if ${}_1\pi_{c,jj} > 0$; E_j is a positive (aperiodical) state of ${}_2M$ if and only if $\lim_{n\to\infty} {}_2p_{jj}^{(n)}(t_0) > 0$. The i assertion now follows by (5.2).

A state $E_j \in \mathscr{E}$ is a transient state of ${}_1M({}_2M)$ if

$$\int_0^\infty {}_1p_{jj}(t)dt < \infty, \quad \sum_{n=0}^{\infty} {}_2p_{jj}^{(n)}(t_0) < \infty, \text{ respectively;}$$

it is a null state (aperiodical) if this integral (sum) diverges and

$$\lim_{t\to\infty} {}_1p_{jj}(t) = 0, \quad \lim_{n\to\infty} {}_2p_{jj}^{(n)}(t_0) = 0, \text{ respectively.}$$

Hence for the proof of the ii. and iii. assertion we have to consider

$$\sum_{n=0}^{\infty} {}_2p_{jj}^{(n)}(t_0),$$

since by (5.2)

$$\lim_{t\to\infty} {}_1p_{jj}(t) = \lim_{n\to\infty} {}_2p_{jj}^{(n)}(t_0).$$

Putting

(5.4) $$\beta^{(m)}(t) \overset{\text{df}}{=} \sum_{n=1}^{m} a^{n*}(t), \quad t \in T_c^{-\infty}, \ m \in N_1,$$

it follows

(5.5) $$\sum_{n=1}^{m} {}_2p_{jj}^{(n)}(t_0) = \int_{-\infty}^{+\infty} {}_1p_{jj}(t)\, d_t\beta^{(m)}(t), \quad m \in N_1.$$

Consider a renewal process described by an infinite sequence of independent stochastic variables all with the same probability distribution $a(.)$, and let the process start at $t = 0$. From renewal theory (cf. [6]) it follows that $\beta^{(m)}(t)$ converges for $m \to \infty$ weakly to the average number $\beta^{(\infty)}(t)$ of renewals in the interval $[0, t]$, $t \in [0, \infty)$, i.e. for any finite t belonging to the continuity set of $\beta^{(\infty)}(.)$

$$\beta^{(\infty)}(t) = \lim_{m \to \infty} \beta^{(m)}(t) < \infty. \tag{5.6}$$

We now prove (cf. (5.5)) that

$$\sum_{n=1}^{\infty} {}_2p_{jj}^{(n)}(t_0) = \int_{-\infty}^{+\infty} {}_1p_{jj}(t)\, d\beta^{(\infty)}(t), \tag{5.7}$$

in the sense that both members are infinite, or both are finite and then equal.

From (5.4) and (5.6) it is seen that $\beta^{(\infty)}(.)$, and $\beta^{(m)}(.)$ for every fixed $m \in N_1$, are nondecreasing functions, moreover ${}_1p_{jj}(.)$ is continuous on $(-\infty, \infty)$ (cf. c.i. section 2) so that by the Helly–Bray lemma (cf. [2] p. 180) for any finite nonnegative number T_1 belonging to the continuity set of $\beta^{(\infty)}(.)$

$$\lim_{m \to \infty} \int_{-T_1}^{T_1} {}_1p_{jj}(t)\, d\beta^{(m)}(t) = \int_{-T_1}^{T_1} {}_1p_{jj}(t)\, d\beta^{(\infty)}(t).$$

Suppose

$$\xi \stackrel{\text{df}}{=} \lim_{m \to \infty} \int_{-\infty}^{+\infty} {}_1p_{jj}(t)\, d\beta^{(m)}(t) < \infty,$$

then

$$\xi \geqq \lim_{m \to \infty} \int_{-T_1}^{T_1} {}_1p_{jj}(t)\, d\beta^{(m)}(t) = \int_{-T_1}^{T_1} {}_1p_{jj}(t)\, d\beta^{(\infty)}(t),$$

so that since the last term is nondecreasing in T_1

$$\xi \geqq \int_{-\infty}^{+\infty} {}_1p_{jj}(t)\, d\beta^{(\infty)}(t).$$

Suppose next

$$\zeta \stackrel{\text{df}}{=} \int_{-\infty}^{+\infty} {}_1p_{jj}(t)\, d\beta^{(\infty)}(t) < \infty;$$

since by (5.4)

$$\int_{-\infty}^{+\infty} {}_1p_{jj}(t) d\beta^{(m)}(t) \leqq \int_{-\infty}^{+\infty} {}_1p_{jj}(t) d\beta^{(m)}(t) + \int_{-\infty}^{+\infty} {}_1p_{jj}(t) d \sum_{h=m+1}^{\infty} a^{h*}(t) = \zeta,$$

it follows that $\xi \leqq \zeta$. Consequently, since $\xi < \infty$ implies $\zeta < \infty$ and conversely (5.7) is proved.

Concerning the convergence or divergence of

$$\int_{-\infty}^{+\infty} {}_1p_{jj}(t)\, d\beta^{(\infty)}(t)$$

it suffices to consider the behaviour of

$$\int_{T_1}^{\infty} {}_1p_{jj}(t)\, d\beta^{(\infty)}(t),$$

since

$$\int_{-\infty}^{T_1} {}_1p_{jj}(t)\, d\beta^{(\infty)}(t) \leqq \beta^{(\infty)}(T_1) < \infty .$$

To find out the behaviour of the last but one integral we need some results from renewal theory.

Putting

$$\alpha_c \stackrel{\mathrm{df}}{=} \{ \int_{-\infty}^{+\infty} t da(t)\}^{-1}, \text{ if the integral is finite,}$$

$$\stackrel{\mathrm{df}}{=} 0 \qquad\qquad , \text{ otherwise,}$$

then renewal theory states that

$$\beta^{(\infty)}(t+\tau) - \beta^{(\infty)}(t) \to \alpha_c \tau \text{ if } t \to \infty, \text{ for all } \tau \in [0, \infty),$$

whenever $a(.)$ is not a lattice distribution. Evidently we have in this case for T_1 sufficiently large

$$\int_{T_1}^{\infty} {}_1p_{jj}(t) d\beta^{(\infty)}(t) \sim \alpha_c \int_{T_1}^{\infty} {}_1p_{jj}(t) dt \text{ if } \alpha_c > 0;$$

i.e. whenever $\alpha_c > 0$

$$\int_{T_1}^{\infty} {}_1p_{jj}(t) d\beta^{(\infty)}(t) < \infty \Leftrightarrow \int_{T_1}^{\infty} {}_1p_{jj}(t) dt < \infty .$$

If $\alpha_c = 0$ then evidently

$$\int_{T_1}^{\infty} {}_1p_{jj}(t) dt < \infty \Rightarrow \int_{T_1}^{\infty} {}_1p_{jj}(t) d\beta^{(\infty)}(t) < \infty ,$$

however, the converse statement is not necessarily true; to reach herefore a definite conclusion more should be known about the behaviour of $\beta^{(\infty)}(t+\tau) - \beta^{(\infty)}(t)$ and of ${}_1p_{jj}(t)$ for $t \to \infty$.

From the results obtained above it is evident that the ii. and iii. assertions are true whenever $a(.)$ is not a lattice distribution.

Consider the case that $a(.)$ is a lattice distribution, so that we may write

$$(5.9) \qquad \begin{cases} a(t) = \sum_{n=0}^{\infty} a_n U(t - nt_0), \ t \in (-\infty, \infty), \\ 0 \leqq a_0 < 1 \text{ (cf. 3.14)}, \ 0 \leqq a_n \leqq 1 \text{ for } n \in N_1, \ \sum_{n=0}^{\infty} a_n = 1. \end{cases}$$

By $\{a_n^{(m)}\}$, $n = 0, 1, 2, \ldots$, $m \in N_1$ we denote for fixed $m \in N_1$ the sequence representing the m-fold convolution of the sequence $\{a_n\}$, $n = 0, 1, 2, \ldots$, with itself, so that

$$(5.10) \qquad a^{m*}(t) = \sum_{n=0}^{\infty} a_n^{(m)} U(t - nt_0), \ t \in (-\infty, \infty).$$

Putting

$$\beta_n^{(m)} \overset{\mathrm{df}}{=} \sum_{h=1}^{m} a_n^{(h)}, \quad n \in N_0, \quad m \in N_1, \tag{5.11}$$

then by (5.6)

$$\beta^{(\infty)}(t) = \sum_{n=0}^{\infty} \beta_n^{(\infty)} U(t-nt_0), \quad t \in (-\infty, \infty),$$

where

$$\beta_n^{(\infty)} \overset{\mathrm{df}}{=} \lim_{m \to \infty} \beta_n^{(m)}, \quad n \in N_0. \tag{5.12}$$

Let $d_1 \geqq 1$ represent the highest common divisor of all those $n \in N_0$ for which $a_n > 0$; then renewal theory states (cf. [7] p. 272)

$$\begin{cases} \beta_n^{(\infty)} = 0 \text{ for } n \neq 0 \bmod d_1, \\ \lim\limits_{n \to \infty} \beta_{nd_1}^{(\infty)} = \alpha_c d_1. \end{cases} \tag{5.13}$$

The relation (5.7) may now be rewritten as

$$\sum_{n=1}^{\infty} {}_2p_{jj}^{(n)}(t_0) = \sum_{n=0}^{\infty} {}_1p_{jj}(nd_1\, t_0)\, \beta_{nd_1}^{(\infty)}. \tag{5.14}$$

Chung proved that $\int_0^{\infty} {}_1p_{jj}(t)dt$ diverges if and only if $\sum_{n=0}^{\infty} {}_1p_{jj}(nh)$ diverges for some $h > 0$, in which case the sum diverges for all $h > 0$ (cf. [1] p. 180). Applying this result the proof of the ii. and iii. assertions follows from (5.13) and (5.14) analogous to the case where $a(.)$ is not a lattice distribution. The proof is terminated.

We shall give an example which shows that actually both cases mentioned in assertion iii. of theorem 5.1 may occur. The original chain shall be the symmetrical random walk with continuous timeparameter, with state space the set of all integers and with Q-matrix defined by

$$\begin{cases} {}_1q_{ij} = 0 \text{ for } |i-j| \neq 1, \ i \neq j, \ i, j \in\in N_{-\infty}, \\ \quad\ = \tfrac{1}{2} \text{ for } |i-j| = 1, \\ \quad\ = 0 \text{ for } \quad i = j. \end{cases} \tag{5.15}$$

The matrix ${}_1P(t)$ of the c.p. Markov chain ${}_1M$ is uniquely determined by the forward system of differential equations

$$\frac{d}{dt}\, {}_1p_{ij}(t) = -\, {}_1p_{ij}(t) + \tfrac{1}{2}\, {}_1p_{i,j-1}(t) + \tfrac{1}{2}\, {}_1p_{i,j+1}(t),$$
$${}_1p_{ij}(0+) = \delta_{ij}, \quad i, j \in\in N_{-\infty}.$$

By reasons of symmetry

$${}_1p_{i,i+k}(t) = {}_1p_{i,i-k}(t), \quad k \in N_0.$$

The solution [1]) of the set of equations above reads

(5.16) $\quad {}_1p_{i,i\pm k}(t) = e^{-t} I_k(t),\ t \in [0, \infty),\ i \in N_{-\infty},\ k \in N_0,$

where

$$I_k(t) = \sum_{m=0}^{\infty} \frac{(\frac{1}{2}t)^{k+2m}}{m!\, \Gamma(k+m+1)},$$

i.e. $I_k(t)$ is the modified Besselfunction of the kth order. Since for large values of t

$$e^{-t} I_k(t) \sim (2\pi t)^{-\frac{1}{2}},$$

it is evident that all states of ${}_1M$ are null states.

As deriving function we take a lattice distribution (cf. (5.9)). We define here $a(\,.\,)$ by the generating function of the sequence $\{a_n\}, n = 0, 1, 2, \ldots$, i.e.

(5.17)
$$\begin{cases} f(\lambda) \overset{\mathrm{df}}{=} \sum_{n=0}^{\infty} a_n \lambda^n, \ \lambda \in [-1, 1), \\ f^m(\lambda) = \sum_{n=0}^{\infty} a_n^{(m)} \lambda^n. \end{cases}$$

To specify for our example the deriving function we take

(5.18) $\quad f(\lambda) \overset{\mathrm{df}}{=} 1 - (1-\lambda^2)^{\delta},\ 0 < \delta < 1,\ \lambda \in [-1, 1),$

and it is easily verified that this function determines according to (5.9) and (5.17) for all $0 < \delta < 1$ a lattice distribution $a(\,.\,) = b(t_0, .)$.

Since, by (5.11), (5.12) and (5.17)

$$\begin{aligned} \sum_{n=0}^{\infty} \beta_n^{(\infty)} \lambda^n &= \frac{f(\lambda)}{1 - f(\lambda)} \text{ for } \lambda \in [-1, 1), \\ &= (1-\lambda^2)^{-\delta} - 1 \text{ by (5.18)}, \end{aligned}$$

it follows

$$\beta_0^{(\infty)} = 0,\ \beta_{2n-1}^{(\infty)} = 0,\ \beta_{2n}^{(\infty)} = (-1)^n \binom{-\delta}{n},\ n \in N_1,$$

so that

$$\beta_{2n}^{(\infty)} \sim \frac{1}{\Gamma(\delta)} \frac{1}{n^{1-\delta}} \text{ for } n \to \infty.$$

Hence, since by (5.16)

$${}_1p_{i,i\pm k}(nt_0) \sim (2\pi n t_0)^{-\frac{1}{2}} \text{ for } n \to \infty,$$

it follows

$$\begin{aligned} {}_2p_{jj}^{(2n+1)}(t_0)\, \beta_{2n+1}^{(\infty)} &= 0 \text{ for all } n \in N_0, \\ {}_2p_{jj}^{(2n)}(t_0)\, \beta_{2n}^{(\infty)} &\sim \frac{1}{\sqrt{2\pi}\, \Gamma(\delta) n^{1\frac{1}{2}-\delta}} \text{ for } n \to \infty. \end{aligned}$$

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[1]) For another derivation of the solution (5.16) see section 10.

Consequently, all states of ${}_2M$ are transient if $0<\delta<\frac{1}{2}$, whereas they are null states if $\frac{1}{2}\leqq\delta\leqq 1$.

Finally, we consider the relation between a closed class of ${}_1M$ and of ${}_2M$.

A class $\mathscr{F}$ of states $E_i \in \mathscr{E}$ is closed for the chain ${}_1M$ if for some $t \in T_c^{0+}$ (and hence all $t \in T_c^{0+}$ cf. c.i.i. section 2)

$$ {}_1p_{ij}(t) = 0 \text{ for all } E_i \in \mathscr{F}, \text{ and all } E_j \notin \mathscr{F}; $$

$\mathscr{F}$ is closed for ${}_2M$ if

$$ {}_2p_{ij}(t_0) = 0 \text{ for all } E_i \in \mathscr{F} \text{ and all } E_j \notin \mathscr{F}; $$

$\mathscr{F}$ is called minimal if $\mathscr{F}$ does not contain a proper subclass which is closed.

Theorem 5.2. Under the conditions of theorem 5.1 if $\mathscr{F}$ is closed for ${}_1M$ then it is closed for ${}_2M$ and conversely; further if $\mathscr{F}$ is minimal for ${}_1M$ then it is minimal for ${}_2M$ and conversely.

Proof. If ${}_1p_{ij}(t)=0$, $i\neq j$, for all $t \in T_c^0$ then by (5.1) ${}_2p_{ij}(nt_0)=0$ for all $n \in N_1$. Conversely, if ${}_2p_{ij}(t_0)=0$, $i\neq j$, then by (5.1) with $n=1$ necessarily ${}_1p_{ij}(t)=0$ for at least one $t \in T_c^{0+}$ and hence for all $t \in T_c^{0+}$. From these statements the assertions of the theorem follow immediately. The proof is terminated.

(To be continued)